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Axisymmetric, Non-ideal MHD States with Steady Flow

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Abstract

Toroidal plasma configurations with steady flow are studied in the framework of non-ideal MHD theory. The properties of the resulting set of equations are examined.
The numerical solution of the two-dimensional, non-linear system appears feasible, although the large variation in the transport coefficients creates considerable numerical problems.

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0. Introduction

Tokamak discharges can be sustained for several seconds with an energy confinement time approaching 100 milliseconds. These experiments are, in general, not terminated by instabilities in the form of a disruption. A suitable description of the plasma behaviour is given by the macroscopic model (MDH). To reproduce the essential features of the experiment, the MHD equations have to incorporate the non-linear dependence and to include non-ideal effects. The full time-dependent problem is, in our opinion, still far too complex for sufficiently accurate numerical treatment. We therefore treat the long-time evolution as an equilibrium problem, where the plasma passes through a sequence of equilibrium states. By imposing axi-symmetry, which is adequate for tokamak configurations, the corresponding problem reduces to determining two-dimensional equilibria. This is feasible with existing numerical techniques and computing facilities.

The ideal MHD model with scalar pressure has been remarkably successful for describing a plasma. Owing to the characteristic Alfvén time scale of the order of microseconds, the plasma cannot be too far from an equilibrium. This feature is built into H. Grad's 1 -1/2 D transport scheme /1/, where the plasma passes through two-dimensional equilibria obyeing the equation $\nabla p = \underline{J} \times \underline{B}$ and the profiles evolve as surface quantities. This model is, however, only adequate if the plasma flow is small and if the pressure and density are, basically, surface quantities.

The large amount of additional heating in the form of neutral beam injection, which is employed in all major tokamak experiments, acts as a source for toroidal flow with a flow velocity approaching the ion sound speed. The induced poloidal flow, on the other hand, does not exceed a certain value, its damping out leading to a poloidal dependence of the pressure and density on magnetic surfaces. The ideal MHD model can easily be extended to include flow. A code for computing such equilibria with flow has been developed by Kerner and Jandl /2/. (Other

contributions and results are referenced in /2/.)
This model, however, neglects dissipative effects and is not sufficiently general with respect to boundary conditions and sources.

If resistivity is taken into account, toroidal equilibria require a flow for their existence, as is pointed out in Ref. /3/. Pfirsch-Schlüter diffusion, Ref. /4/, sustains such a pressure-driven flow. For ohmic discharges these flows are usually small. But the additional heating drastically enlarges the flow. The viscosity is just as important, and so is the energy flux due to temperature gradients. Taking into account the resistivity, viscosity and heat conductivity yield a set of equations which allow a realistic simulation of an experiment. The hyperbolic character of the continuity equation requires sources for its solution. The remaining set forms an elliptic set. The requirement of an elliptic characteristic is an important point in our analysis since for mixed systems we expect tremendous numerical difficulties. The essential role of the continuity equation is then apparent. The choice of boundary data and source terms should select certain equilibria and forbid others. A close connection with experimental data is possible. We thus can hope for results explaining the density limit being observed in the experiments.

For our macroscopic model transport coefficients are required. In the collision-dominated regime the coefficients are explicitly known. More appropriate is the neoclassical regime, where the trapping of particles is taken into account. The coefficients have different values along and perpendicular to the magnetic field. The consequences of this anisotropy with respect to the numerical approximation are discussed.

The basic assumption in this paper is that the additional heating deposited in the plasma in the form of neutral beam injection and wave heating makes a two-dimensional treatment of the equilibrium problem necessary - owing to the poloidal variation of important quantities such as pressure and density.

The physical model, its mathematical properties and the consequences for the numerical solution are analyzed. The single-fluid model containing non-linear and non-ideal features exhibits such a complexity that a detailed discussion of its properties - especially with respect to a numerical approximation - is both useful and necessary.

We are aware of the fundamental role of the transport coefficients for our model. Experimental data show that some coefficients differ from their classical or neoclassical values and cause anomalous transport. A very interesting aspect of our model is therefore its potential for determining the transport coefficients.

The paper is organized as follows:
Section I presents the fluid equations used throughout the analysis. In Sect. II the constraint of incompressibility is incorporated into this model, and the equations are derived and discussed. The non-ideal, compressible fluid is treated in Sect. III. After the energy equation, the validity of the incompressibility assumption is discussed. The general pressure tensor is derived in Sect. IV, with only covariance and symmetry properties being used. Such a derivation is useful for understanding the macroscopic features of this tensor, especially if one aims to use a simpler form instead of the full tensor. The transport coefficients are listed in Sect. V. The discussion and the conclusions are then finally presented in Sect. VI.

I. Fluid Equations

To begin with, we list the MHD equations for a single-fluid theory. This model relates the density ϱ , velocity $\underline{u}$, scalar pressure p, pressure tensor $\underline{\underline{P}}$, internal energy e and the magnetic field $\underline{B}$:

(1) Continuity: $$\frac{\partial}{\partial t}\varrho + \nabla\cdot(\varrho\underline{u}) = \varrho_s$$

(2) Momentum: $$m\varrho\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)\underline{u} = -\nabla p + \underline{j}\times\underline{B} + \nabla\cdot\mathbf{P}$$

(3) Energy $$\varrho\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)e = -\nabla\underline{q} + \underline{j}\cdot(\underline{E} + \underline{u}\times\underline{B}) + \underline{\underline{P}} : \underline{\nabla}\,\underline{u} + Q_s$$

where $\underline{q}$ denotes the heat flux and ϱ_s and Q_s mass, resp. energy sources

(4) Ohm's law: $$\underline{\underline{\eta}}\,\underline{j} + \eta_H\,\underline{j}\times\underline{B} = \underline{E} + \underline{u}\times\underline{B},$$

where $\underline{\underline{\eta}}$ is the resistivity tensor, η_H the Hall constant, $\underline{E}$ the electric field and $\underline{J}$ the current.

(5) Maxwell: $$\frac{\partial\underline{B}}{\partial t} = -\nabla\times\underline{E}$$

$$\underline{j} = \nabla\times\underline{B}$$

$$\nabla\cdot\underline{B} = 0$$

By means of the thermodynamic relations the pressure is derived from the internal energy and entropy S:

(6) $$p = -\left(\frac{\partial e}{\partial \tau}\right)_S \qquad \tau = \frac{1}{\varrho}$$

II. Incompressible Fluid

The fluid equations for stationary, axisymmetric configurations are discussed in usual cylindrical coordinates r, θ, z, where θ is the ignorable coordinate. The equations are also given in planar symmetry with z as the ignorable coordinate.
Introducing the condition of incompressibility

(7) $$\nabla \cdot \underline{u} = 0$$

replaces the energy equation (3). The pressure p can be eliminated to avoid the thermodynamic relation, e.g. $p = p(e, S)$ or $p = p(\rho, T)$, in the equilibrium calculation.
Equation (7) suggests the introduction of a stream function λ for the velocity:

(8) $$\underline{u} = \nabla\lambda \times \nabla\theta + \omega \nabla\theta$$

and in planar geometry

(8') $$\underline{u} = \nabla\lambda \times \nabla z + \omega \nabla z$$

The equation of continuity (1) reads

$$\underline{u} \cdot \nabla\rho = \nabla\lambda \times \nabla\theta \cdot \nabla\rho = 0$$

implying that the density ρ is a function of λ only:

(9) $$\rho = \rho(\lambda)$$

The condition that the magnetic field is divergence-free is satisfied by introducing the poloidal flux function Ψ and the poloidal current profile χ:

(10) $$\underline{B} = \nabla\Psi \times \nabla\theta + \chi \nabla\theta$$

or

(10') $$\underline{B} = \nabla\Psi \times \nabla z + \chi \nabla z$$

Ampère's law defines the current:

$$\underline{j} = \nabla \times \underline{B}$$

Using the relation

$$\nabla\theta \cdot \nabla \times (\nabla\psi \times \nabla\theta) = \nabla[(\nabla\psi \times \nabla\theta) \times \nabla\theta] = -\nabla \cdot (\nabla\psi \cdot |\nabla\theta|^2)$$

we obtain

(11) $$\underline{j} = -\nabla\theta \times \nabla\chi - \nabla\theta\, r^2\, \nabla \cdot (\nabla\psi / r^2) = -\nabla\theta \times \nabla\chi - \nabla\theta\, j_0$$

or

(11') $$\underline{j} = -\nabla z \times \nabla\chi - \nabla z\, \Delta\psi \qquad = -\nabla z \times \nabla\chi - \nabla z\, j_0$$

Maxwell's equation $\nabla \times E = 0$ is combined with Ohm's law, which states a relation between the current and electric field, eq. (4). Only a scalar resistivity is taken into account here:

(12) $$\eta\, \underline{j} = \underline{E} + \underline{u} \times \underline{B}$$

We have

$$\nabla \times \underline{E} = \nabla \times (\eta \underline{j} - \underline{u} \times \underline{B})$$

(13) $$= -\nabla \times (\eta\, \nabla\theta \times \nabla\chi) + \nabla\theta \times \nabla[\eta j_0 - \nabla\psi \times \nabla\lambda \cdot \nabla\theta]$$
$$- \nabla\lambda \times \nabla\chi / r^2 + \nabla\psi \times \nabla(\omega / r^2)\ ,$$

with J_o defined in eq. (11).

Evaluating the θ-component of this equation, we obtain

(14) $$\nabla \cdot (\eta / r^2\ \nabla\chi) = -\nabla\lambda \times \nabla(\chi / r^2) \cdot \nabla\theta + \nabla\psi \times \nabla(\tfrac{\omega}{r^2}) \cdot \nabla\theta.$$

Next we operate with the vector product $\nabla\theta\ \times$ on eq. (13), and using the relation

$$\nabla\theta \times \nabla \times (\eta\, \nabla\theta \times \nabla\chi) = \nabla\theta \times [\eta \nabla \times (\nabla\theta \times \nabla\chi) - (\nabla\theta \times \nabla\chi) \times \nabla\eta] = 0$$

we obtain

(15) $$\nabla[\eta j_0 - \nabla\psi \times \nabla\lambda \cdot \nabla\theta] = 0$$

and, furthermore,

(16) $$\eta r^2 \nabla\cdot(\nabla\psi/r^2) = r_0 E_0 + \nabla\psi \times \nabla\lambda\cdot\nabla\theta ,$$

where $r_o E_o$ is the constant toroidal electric field at $r = r_o$.

We now address the momentum equation (2)

$$m\varrho\, \underline{u}\cdot\nabla\underline{u} + \nabla p - \nabla\cdot\underline{\underline{P}} - \underline{j}\times\underline{B} = 0$$

To simplify the discussion, we begin with a scalar viscosity μ and take the stress tensor from ordinary fluid theory:

(17) $$\underline{\underline{P}} = \mu\, \underline{\underline{W}}$$

with

(18) $$W_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\,\nabla\cdot\underline{u}$$

In general coordinates the derivatives have to be replaced by covariant derivatives. In cylindrical coordinates the stress tensor for an incompressible fluid with axisymmetry is of the form

(19) $$\begin{aligned}
W_{rr} &= 2u_{r,r} &&= 2\left(-\tfrac{1}{r}\lambda_{,zr} + \lambda_{,z}/r^2\right)\\
W_{r\theta} &= u_{\theta,r} - \frac{u_\theta}{r} &&= \left(\frac{\omega}{r}\right)_{,r} - \frac{\omega}{r^2}\\
W_{rz} &= u_{r,z} + u_{z,r} &&= -\frac{\lambda_{,zz}}{r} + \frac{\lambda_{,rr}}{r} - \frac{\lambda_{,r}}{r^2}\\
W_{\theta\theta} &= 2\frac{u_r}{r} &&= -2\frac{\lambda_{,z}}{r^2}\\
W_{\theta z} &= u_{\theta,z} &&= \frac{\omega_{,z}}{r}\\
W_{zz} &= 2u_{z,z} &&= 2\lambda_{,rz}/r
\end{aligned}$$

where $\lambda_{,r}$ denotes the partial derivative $\frac{\partial\lambda}{\partial r}$, etc.

The divergence of the pressure tensor is evaluated by writing

(20) $$\underline{\underline{\mathbb{P}}} = \sum_{k=1}^{3} \underline{i}_k \, \underline{\mathbb{P}}_k$$

where the $\underline{i}_k$ denote the unit vectors.
This gives

(21) $$\nabla \cdot \underline{\underline{\mathbb{P}}} = \sum_k \left\{ \underline{i}_k \, \nabla \cdot \underline{\mathbb{P}}_k + (\underline{\mathbb{P}}_k \cdot \nabla) \, \underline{i}_k \right\}$$

$$= \underline{i}_r \left(\nabla \cdot \underline{\mathbb{P}}_r - \mathbb{P}_{\theta\theta}/r \right) + \underline{i}_\theta \left(\nabla \cdot \underline{\mathbb{P}}_\theta + \mathbb{P}_{r\theta}/r \right)$$

$$+ \underline{i}_z \, \nabla \cdot \underline{\mathbb{P}}_z$$

We get with

$$\nabla \cdot \underline{\mathbb{P}}_r = \frac{1}{r} \left\{ (r \mathbb{P}_{rr})_{,r} + (r \mathbb{P}_{rz})_{,z} \right\}$$

the result

(22)
$$\nabla \cdot \underline{\mathbb{P}}_r = \nabla \cdot (\mu \, \nabla u_r) - \mu \, \nabla \cdot \left(\frac{\underline{u}}{r}\right) + \underline{u}_{,r} \cdot \nabla \mu ,$$

$$\nabla \cdot \underline{\mathbb{P}}_\theta = \nabla \cdot (\mu \, \nabla u_\theta) - \mu \, \frac{u_{\theta,r}}{r} - u_\theta \, \frac{\mu_{,r}}{r} ,$$

$$\nabla \cdot \underline{\mathbb{P}}_z = \nabla \cdot (\mu \, \nabla u_z) + \underline{u}_{,z} \cdot \nabla \mu$$

Next we treat the term $\underline{u} \cdot \nabla \underline{u}$ and use the relation

(23) $$(\underline{u} \cdot \nabla) \, \underline{u} = \frac{1}{2} \nabla (\underline{u}^2) - \underline{u} \times \nabla \times \underline{u}$$

With the help of

$$\nabla \times \underline{u} = - \nabla \theta \, r^2 \, \nabla \cdot (\nabla \lambda / r^2) - \nabla \theta \times \nabla \omega$$

and

$$\underline{u} \times \nabla \times \underline{u} = \nabla \lambda \;\; \nabla \cdot (\nabla \lambda / r^2) - \nabla \theta \;\; \nabla \omega \times \nabla \lambda \cdot \nabla \theta + \frac{\omega}{r^2} \nabla \omega$$

we get the expression

$$\underline{\underline{u}}\cdot\nabla\underline{u} = \nabla\left[|\nabla\lambda|^2/2r^2 + \omega^2/2r^2\right] - \nabla\cdot(\nabla\lambda/r^2)\,\nabla\lambda \tag{24}$$
$$- \frac{\omega^2}{r^2}\nabla\omega - \nabla\lambda\times\nabla\omega\cdot\nabla\theta\;\nabla\theta .$$

The Lorentz force is given by

$$\underline{j}\times\underline{B} = -\frac{j_0}{r^2}\nabla\psi - \frac{\chi}{r^2}\nabla\chi + \nabla\theta\;\nabla\chi\times\nabla\psi\cdot\nabla\theta. \tag{25}$$

The momentum equation eventually assumes the form

$$mg\,\nabla\left[|\nabla\lambda|^2/2r^2 + \frac{\omega^2}{2r^2}\right] - mg\,\nabla\cdot(\nabla\lambda/r^2)\,\nabla\lambda - mg\,\omega/r^2\,\nabla\omega$$
$$-mg\nabla\theta\,\nabla\lambda\times\nabla\omega\cdot\nabla\theta + \frac{j_0}{r^2}\nabla\psi + \frac{\chi}{r^2}\nabla\chi + \nabla\psi\times\nabla\chi\cdot\nabla\theta\;\nabla\theta + \nabla p =$$
$$= \underline{i}_r\left[\nabla\cdot(\mu\nabla u_r) - \mu\,\nabla\cdot\left(\frac{u}{r}\right) + \underline{\underline{u}}_{,r}\cdot\nabla\mu + \frac{2\mu}{r^3}\lambda_{,z}\right] + \tag{26}$$
$$+\,\underline{i}_\theta\left[\nabla\cdot(\mu\nabla u_\theta) - \frac{u_\theta}{r}\left(\mu_{,r} + \frac{\mu}{r}\right)\right] +$$
$$+\,\underline{i}_z\left[\nabla\cdot(\mu\nabla u_z) + \underline{\underline{u}}_{,z}\cdot\nabla\mu\right].$$

The θ-component of this equation reads:

$$r\nabla(mg\omega)\times\nabla\lambda\cdot\nabla\theta + r\nabla\psi\times\nabla\chi\cdot\nabla\theta = r^2\,\nabla\cdot\left(\mu\,\nabla\left(\frac{\omega}{r}\right)\right) - \tag{27}$$
$$-\,\omega\left(\mu_{,r} + \frac{\mu}{r}\right).$$

The r-component is of the form

$$\nabla\lambda\times\nabla(mg\,\lambda_{,z}/r)\cdot\nabla\theta - mg\,\frac{\omega^2}{r^3} + \frac{j_0}{r^2}\psi_{,r} + \left(p + \frac{\chi^2}{2r^2}\right)_{,r} + \frac{\chi^2}{r^3} = \tag{28}$$
$$= -\nabla\cdot\mu\nabla\left(\frac{\lambda_{,z}}{r}\right) + \nabla\mu\times\nabla\lambda_{,r}\cdot\nabla\theta -$$
$$-\,\frac{1}{r}\nabla\mu\times\nabla\lambda\cdot\nabla\theta + \mu\,\lambda_{,z}/r^3 ,$$

and the z-component

$$(29) \qquad -\nabla\lambda \times \nabla\left(mg\,\frac{\lambda_{,r}}{r}\right)\cdot\nabla\theta + \frac{j_0}{r^2}\,\psi_{,z} + \left(p + \frac{\chi^2}{2r^2}\right)_{,z} =$$

$$= \nabla\cdot\mu\,\nabla\left(\frac{\lambda_{,r}}{r}\right) + \nabla\mu \times \nabla\lambda_{,z}\cdot\nabla\theta$$

The momentum equation contains the term ∇p. Usually the energy equation determines the internal energy e (or the entropy S or the temperature T) of the system. However, the incompressibility constraint (7) allows the energy equation to be neglected. The pressure is eliminated from the momentum equation and can be determined if $\underline{u}$ and $\underline{B}$ are known. For this purpose we differentiate eq. (29) with respect to r and eq. (28) with respect to z and subtract the resulting equations from each other. Finally, we summarize all the equations using the summation convention with respect to the indices α and β and the definition $\epsilon_{\alpha\beta} = 1$ if $\alpha = r$ and $\beta = z$, and -1 if $\alpha = z$ and $\beta = r$, and end up with the system (S1):

$$(30) \qquad \nabla\cdot\left(\frac{M}{r^2}\,\nabla\chi\right) = -\nabla\lambda \times \nabla\left(\frac{\chi}{r^2}\right)\cdot\nabla\theta + \nabla\psi \times \nabla\left(\frac{\omega}{r^2}\right)\cdot\nabla\theta$$

$$(31) \qquad r^2\,\eta\,\nabla\cdot\left(\nabla\psi/r^2\right) = r_0 E_0 + \nabla\psi \times \nabla\lambda\cdot\nabla\theta$$

$$(32) \qquad r\,\nabla\left(mg\,\omega\right) \times \nabla\lambda\cdot\nabla\theta + r\,\nabla\psi \times \nabla\chi\cdot\nabla\theta =$$

$$= r^2\,\nabla\cdot\,\mu\,\nabla\left(\frac{\omega}{r}\right) - \omega\left(\mu_{,r} + \frac{\mu}{r}\right),$$

$$(33) \qquad \partial_\alpha\,\nabla\left(mg\,\lambda_{,\alpha}/r\right) \times \nabla\lambda\cdot\nabla\theta =$$

$$= \partial_\alpha\,\nabla\cdot\mu\,\nabla\left(\lambda_{,\alpha}/r\right) + \epsilon_{\alpha\beta}\,\nabla\mu_{,\alpha} \times \nabla\lambda_{,\beta}\cdot\nabla\theta +$$

$$+ \frac{1}{r}\,\nabla\mu_{,z} \times \nabla\lambda\cdot\nabla\theta - \frac{1}{r^3}\left(\mu\,\lambda_{,z} - \chi^2 + mg\,\omega^2\right)_{,z} +$$

$$+ \nabla\left(\frac{j_0}{r^2}\right) \times \nabla\psi\cdot\nabla\theta.$$

In planar symmetry the system equations are of the form:

(30') $\nabla\cdot(\eta\nabla\chi) = -\nabla\lambda\times\nabla\chi\cdot\nabla z + \nabla\psi\times\nabla\omega\cdot\nabla z,$

(31') $-\eta\,\Delta\psi = E_0 + \nabla\psi\times\nabla\lambda\cdot\nabla z,$

(32') $\nabla(mg\omega)\times\nabla\lambda\cdot\nabla z + \nabla\psi\times\nabla\chi\cdot\nabla z = \nabla\cdot\mu\nabla\omega,$

(33') $\partial_\alpha\nabla(mg\lambda_{,\alpha})\times\nabla\lambda\cdot\nabla z = \partial_\alpha\nabla\cdot\mu\nabla\lambda_{,\alpha} - \epsilon_{\alpha\beta}\nabla\mu_{,\alpha}\times\nabla\lambda_{,\beta}\cdot\nabla z$
$+ \nabla(\Delta\psi)\times\nabla\psi\cdot\nabla z$

$(\alpha,\beta = x,y)$

If there is no flow, then it holds that $\lambda\equiv 0$ and $\omega\equiv 0$. From eq. (32) it follows that $\chi = \chi(\psi)$ and eq. (26) reduces to the Grad-Shafranov equation:

(34) $-p_{,\psi} - \chi\chi_{,\psi}/r^2 = \nabla\cdot(\nabla\psi/r^2)\ .$

The system (30' - 33') decouples into two pairs of equations. The functions ψ and λ are determined by eqs. (31') and (33'). Once ψ and λ are known, the functions χ and ω are the solutions of the linear system 30' and 32'. In the toroidal system (30 - 33) the functions χ and ω also occur in eq. (33). If χ and ω are known, the functions ψ and λ are determined by eqs. (31) and (33). This defines an iteration scheme for the numerical solution.

We examine the equations for the poloidal quantities λ and ψ with given ω and χ for the toroidal case, eqs. (31) and (33). For non-zero viscosity, $\mu\neq 0$, this is a fourth-order elliptic equation for λ , together with a second-order elliptic equation for ψ . For this system only boundary data can be prescribed and no profiles can be imposed, except the dependence $g = g(\lambda)$ on the continuity equation. The other two equations for the toroidal quantities ω and χ , eqs. (30) and (32), with given λ and ψ also form an elliptic set for ω and χ .

The case with purely toroidal flow, i.e. $\lambda \equiv 0$, is possible if $r_0 E_0 = \eta j_0 = \eta r^2 \nabla \cdot (\nabla\psi / r^2)$, which determines ψ . Ohm's law for χ, eq. (30), and eq. (32) form a closed set for ω, χ and ψ . Now ϱ is no longer a function of λ , but is arbitrary. The pressure balance, eq. (26), where

$$(35)\quad \nabla\left(p + \frac{\chi^2}{2r^2}\right) + \frac{j_0}{r^2}\nabla\psi + \nabla\left(\frac{1}{r^2}\right)\left(\frac{m}{2}\omega^2\varrho - \chi^2\right) = 0$$

is then highly degenerate. The two components of this equation determine the remaining thermodynamic unknowns ϱ and p . If there is a solution with very small but non-zero toroidal flow ω , its streamlines are, presumably, curves ϱ = ct. of the above solution. It is by no means clear that with viscosity such flows exist since the momentum imbalance may act as a source of poloidal rotation even if no rotation is imposed at the wall.

For zero viscosity, $\mu \equiv 0$, the system has quite different properties. The momentum equation (33) then reduces to

$$(36)\quad \partial_\alpha \nabla\left(m\varrho\lambda_{,\alpha}/r\right) \times \nabla\lambda \cdot \nabla\theta =$$

$$= -\frac{\lambda}{r^3}\left(m\varrho\omega^2 - \chi^2\right)_{,z} + \nabla\left(\frac{j_0}{r^2}\right) \times \nabla\psi \cdot \nabla\theta$$

The solvability condition for this equation is less clear. For given ψ one has to solve a hyperbolic equation for $\nabla\lambda$. This implies that on closed field lines there is a solvability constraint. The free data on $\nabla\lambda$ are needed for this constraint.

The discussion of the energy equation in the next section makes it clear that the assumption of incompressibility is not satisfied for equilibria with flow. On the other hand, the system (30 - 33) is quite complicated. We therefore conclude that the considerable effort for its numerical solution may not be worthwhile.

III. Compressible Fluid

In this section the energy equation (3) is included and the assumption of incompressibility is given up. At first the general Ohm's law, eq. (4), is discussed. The resistivity tensor $\underline{\underline{\eta}}$ is defined as

$$(37) \qquad \underline{\underline{\eta}} = \eta_\perp \mathbb{I} - (\eta_\perp - \eta_\parallel)\, \underline{b}\,\underline{b} ,$$

where $\mathbb{I}$ is the unit matrix, $\underline{b} = \underline{B}/|B|$ the unit vector in the direction of the magnetic field and $\eta_\parallel$ and $\eta_\perp$ the values of the resistivity along and perpendicular to the magnetic field. This implies that

$$(38) \qquad \underline{\underline{\eta}} \cdot \underline{j} = \eta_\perp (\underline{j} - \underline{j}_\parallel) + \eta_\parallel \underline{j}_\parallel = \eta_\perp \underline{j}_\perp + \eta_\parallel \underline{j}_\parallel$$

where $\underline{j}_\parallel = (\underline{j} \cdot \underline{b})\, \underline{b}$
is the component of the current parallel to the magnetic field and $\underline{J}_\perp$ the perpendicular one.
Using eq. (11) for the current yields

$$(39) \qquad \underline{j}_\parallel = (\nabla\psi \cdot \nabla\chi - j_0 \chi)/r^2 \underline{B}^2 \;\; \underline{B} = \tilde{j}_\parallel \underline{B}$$

With the introduction of the electric field

$$\underline{E} = -\nabla\Phi + r_0 E_0 \nabla\theta ,$$

where Φ is the scalar potential (see eq. (16)), Ohm's law takes the form

$$(40) \qquad \eta_\perp \underline{j} + (\eta_\parallel - \eta_\perp)\underline{j}_\parallel + \eta_H \underline{j} \times \underline{B} = -\nabla\Phi + r_0 E_0 \nabla\theta + \underline{u} \times \underline{B} .$$

The projection onto $\nabla\theta$ yields

$$(41) \qquad -j_0 \left[\eta_\perp + (\eta_\parallel - \eta_\perp) \frac{\chi^2}{r^2 \underline{B}^2} \right] + (\eta_\parallel - \eta_\perp)\chi \frac{\nabla\psi \cdot \nabla\chi}{r^2 \underline{B}^2} =$$

$$= r_0 E_0 - \underline{u} \cdot \nabla\psi + \eta_H \nabla\psi \times \nabla\chi \cdot \nabla\theta .$$

Next we examine the $\nabla\theta$ - component of the Maxwell equation $\nabla \times E = 0$:

$$\nabla\theta \cdot \nabla \times \underline{u} \times \underline{B} = \nabla\theta \cdot \nabla \times \left[\eta_\perp \underline{j} + (\eta_\parallel - \eta_\perp)\underline{j}_\parallel + \eta_H \underline{j} \times \underline{B}\right],$$

which is equivalent to

$$\nabla \cdot \nabla\theta \times \underline{u} \times \underline{B} = \nabla \cdot \left\{\eta_\perp \nabla\theta \times \underline{j} + (\eta_\parallel - \eta_\perp) \nabla\theta \times \underline{j}_\parallel + \eta_H \nabla\theta \times \underline{j} \times \underline{B}\right\}$$

and, furthermore,

$$\nabla \cdot \frac{\chi}{r^2} \underline{u} + \nabla\psi \times \nabla\left(\frac{\omega}{r^2}\right) \cdot \nabla\theta =$$

$$= \nabla \cdot \left\{ \frac{\eta_\perp}{r^2} \nabla\chi + (\eta_\parallel - \eta_\perp)/r^2 \, \nabla\psi \, (\nabla\psi \cdot \nabla\chi - \chi j_0)/r^2 \underline{B}^2 \right.$$

$$\left. - \eta_H \left(\frac{j_0}{r^2} \nabla\theta \times \nabla\psi + \frac{\chi}{r^2} \nabla\theta \times \nabla\chi \right\}.$$

We utilize eq. (41) to eliminate j_0 and with

$$\nabla\psi \cdot \nabla\chi - \chi j_0 = \frac{r^2 \underline{B}^2 \left[\eta_\perp \nabla\psi \cdot \nabla\chi + \chi (r_0 E_0 - \underline{u} \cdot \nabla\psi + \eta_H \nabla\psi \times \nabla\chi \cdot \nabla\theta)\right]}{\eta_\perp |\nabla\psi|^2 + \eta_\parallel \chi^2}$$

we eventually get the result

$$\nabla \cdot \left(\frac{\chi}{r^2} \nabla\psi\right) + \nabla\psi \times \nabla\left(\frac{\omega}{r^2}\right) \cdot \nabla\theta = \tag{42}$$

$$= \nabla \cdot \left\{ \frac{\eta_\perp}{r^2} \nabla\chi + (\eta_\parallel - \eta_\perp)/r^2 \, \nabla\psi \, \frac{\eta_\perp \nabla\psi \cdot \nabla\chi + \chi (r_0 E_0 - \underline{u} \cdot \nabla\psi)}{\eta_\perp |\nabla\psi|^2 + \eta_\parallel \chi^2} \right.$$

$$+ \eta_H \nabla\theta \times \nabla\psi / r^2 \, \frac{(r_0 E_0 - \underline{u} \cdot \nabla\psi + \eta_H \nabla\theta \times \nabla\psi \cdot \nabla\chi) \, r^2 \underline{B}^2}{\eta_\perp |\nabla\psi|^2 + \eta_\parallel \chi^2}$$

$$\left. + \frac{\eta_H \eta_\parallel (|\nabla\psi|^2 + \chi^2) \, \chi / r^2 \, \nabla\chi \times \nabla\theta}{\eta_\perp |\nabla\psi|^2 + \eta_\parallel \chi^2} \right\}.$$

Note that at most second derivates of Ψ occur and that all the χ dependence is explicit. It is by no means obvious that eq. (42) is elliptic. The characteristics are clearly those of the equation

$$\eta_\perp \left\{ \Delta\chi + \left[(\eta_\parallel - \eta_\perp) + \eta_H^2\, \underline{B}^2/\eta_\perp\right] (\underline{B}_p\cdot\nabla)^2\chi \Big/ (\eta_\perp \underline{B}_p^2 + \eta_\parallel \underline{B}_t^2) \right\},$$

where $\underline{B}_p$ and $\underline{B}_t$ denote the poloidal and toroidal components of the magnetic field. If we set $k_r = \frac{\partial}{\partial r}$ and $k_z = \frac{\partial}{\partial z}$, the characteristics are

$$k_r^2 + k_z^2 + (\underline{k}\cdot\underline{B}_p)^2 \, \frac{\eta_\parallel - \eta_\perp + \eta_H^2/\eta_\perp \, \underline{B}^2}{\eta_\perp \underline{B}_p^2 + \eta_\parallel \underline{B}_t^2}$$

or

$$\frac{1}{\eta_\perp \underline{B}_p^2 + \eta_\parallel \underline{B}_t^2} \left\{ \eta_\perp \left[\underline{k}^2 \underline{B}_p^2 - (\underline{k}\cdot\underline{B}_p)^2\right] + \eta_\parallel \left[\underline{k}^2 \cdot \underline{B}_t^2 + (\underline{k}\cdot \underline{B}_p)^2\right] + \right.$$
$$\left. + \frac{\eta_H^2}{\eta_\perp} \underline{B}^2 \right\}.$$

Ellipticity is equivalent to the positivity of the quantity

(43) $$\eta_\perp \left[\underline{k}^2 \underline{B}_p^2 - (\underline{k}\cdot\underline{B}_p)^2\right] + \eta_\parallel \left[\underline{k}^2 B_t^2 + (\underline{k}\cdot\underline{B}_p)^2\right] +$$
$$+ \frac{\eta_H^2}{\eta_\perp} \underline{B}^2 .$$

Hence for any $\eta_\perp \geq 0$ and $\eta_\parallel \geq 0$ the system is elliptic.

Next we deal with the momentum equation (2). Writing the velocity $\underline{u}$

(44) $$\underline{u} = \mu \nabla r + v \nabla z + \omega \nabla\theta$$

yields the divergence in the form

(45) $$\nabla\cdot\underline{u} = \frac{1}{r}\left[(r\mu)_{,r} + (rv)_{,z}\right].$$

The stress tensor is of the form according to eqs. (19 - 20):

$$
(46)\qquad
\begin{aligned}
W_{rr} &= 2u_{,r} - \frac{2}{3}\,\nabla\cdot\underline{u} \\
W_{r\theta} &= \left(\frac{\omega}{r}\right)_{,r} - \frac{\omega}{r^2} \\
W_{rz} &= u_{,z} + v_{,r} \\
W_{\theta\theta} &= \frac{2u}{r} - \frac{2}{3}\,\nabla\cdot\underline{u} \\
W_{\theta z} &= \omega_{,z}/r \\
W_{zz} &= 2v_{,z} - \frac{2}{3}\,\nabla\cdot\underline{u}\,.
\end{aligned}
$$

Making use of the expressions from the previous section, we obtain for the r-component of $\nabla\cdot\mathbb{P}$

$$
(47)\qquad
\begin{aligned}
F_r = {} & \nabla\cdot\mu\nabla u + \frac{\lambda}{3}\left(\mu\,\nabla\cdot\underline{u}\right)_{,r} + r\,\nabla\mu\times\nabla v\cdot\nabla\theta \\
& - \mu_{,r}\,\frac{u}{r} - \mu\,\frac{u}{r^2}\quad ,
\end{aligned}
$$

for the θ-component

$$
(48)\qquad F_\theta = \nabla\cdot\mu\,\nabla\left(\frac{\omega}{r}\right) - \mu_{,r}\,\frac{\omega}{r^2} - \mu\,\frac{\omega}{r^3}
$$

and for the z-component

$$
(49)\qquad
\begin{aligned}
F_z = {} & \nabla\cdot\mu\nabla v + \frac{\lambda}{3}\left(\mu\,\nabla\cdot\underline{u}\right)_{,z} - r\,\nabla\mu\times\nabla u\cdot\nabla\theta - \\
& \mu_{,z}\,\frac{u}{r}\quad .
\end{aligned}
$$

The term $\underline{u}\cdot\nabla\underline{u}$ involves the derivatives of the basis vectors

$$
(50)\qquad \underline{u}\cdot\nabla\underline{u} = \nabla r\left[\underline{u}\cdot\nabla u - \frac{\omega^2}{r^3}\right] + \nabla\theta\;\underline{u}\cdot\nabla\omega + \nabla z\;\underline{u}\cdot\nabla v.
$$

The momentum equation is then

$$
(51)\qquad m\varrho\,\underline{u}\cdot\nabla\underline{u} = -\nabla p - \frac{j_0}{r^2}\nabla\psi - \frac{\chi}{r^2}\nabla\chi + \nabla\theta\;\nabla\chi\times\nabla\psi\cdot\nabla\theta + \nabla\cdot\underline{\underline{\mathbb{P}}}\,.
$$

The r-component reads

$$m\rho\left(\underline{u}\cdot\nabla u - \frac{\omega^2}{r^3}\right) + p_{,r} + \frac{1}{r^2}\left(\frac{\chi^2}{2}\right)_{,r} + j_0\,\frac{\psi_{,r}}{r^2} =$$

(52) $$= \nabla\cdot\mu\nabla u + \frac{1}{3}\left(\mu\,\nabla\cdot\underline{u}\right)_{,r} + r\,\nabla\mu\times\nabla v\cdot\nabla\theta -$$

$$-\mu_{,r}\,\frac{u}{r} - \mu\,\frac{u}{r^2},$$

the θ-component

(53) $$m\rho\,\underline{u}\cdot\nabla\omega + \nabla\psi\times\nabla\chi\cdot\nabla\theta = r\,\nabla\mu\cdot\nabla\left(\frac{\omega}{r}\right) - \mu_{,r}\,\frac{\omega}{r} - \mu\,\frac{\omega}{r^2},$$

and the z-component

$$m\rho\,\underline{u}\cdot\nabla v + \left(p + \frac{\chi^2}{2r^2}\right)_{,z} + j_0\,\frac{\psi_{,z}}{r^2} =$$

(54) $$= \nabla\cdot\mu\nabla v + \frac{1}{3}\left(\mu\,\nabla\cdot\underline{u}\right)_{,z} - r\,\nabla\mu\times\nabla u\cdot\nabla\theta - \mu_{,z}u\;/r$$

Clearly this system is elliptic for non-zero viscosity.

Finally, we examine the energy equation. Introducing the temperature instead of the internal energy, we obtain the following energy equation:

$$\frac{3}{2}\rho\,\frac{\partial T}{\partial t} + \frac{3}{2}\rho\,\underline{u}\cdot\nabla T + \rho T\,\nabla\cdot\underline{u} =$$

(55) $$= -\nabla\underline{q} + \underline{j}\left(\underline{E} + \underline{u}\times\underline{B}\right) + \underline{\underline{\mathbb{P}}}:\nabla\underline{u}$$

The heat flux is related to the temperature

(56) $$-\underline{q} = \underline{\underline{\kappa}}\cdot\nabla T = \kappa_{\parallel}\,\nabla_{\parallel}T + \kappa_{\perp}\,\nabla_{\perp}T - \kappa_{\wedge}\,\underline{b}\times\nabla T$$

where parallel $\kappa_{\parallel}$ and perpendicular $\kappa_{\perp}$ and $\kappa_{\wedge}$ are thermal conductivities. The energy flux due to the viscosity is given by

$$\underline{\underline{\mathbb{P}}}:\nabla\underline{u} = \sum_{\alpha,\beta=1,2,3} \mathbb{P}_{\alpha\beta}\,\frac{\partial u_\alpha}{\partial x_\beta}$$

Inserting the terms of the pressure tensor yields

$$\mathbb{P}:\nabla\underline{u} = \mu\Big\{2(u_{,r}^2 + v_{,z}^2) - \frac{2}{3}(u_{,r} + v_{,z})^2 - \tag{57}$$
$$- \frac{2}{3}\frac{u}{r}(u_{,r} + v_{,z}) + (u_{,z} + v_{,r})^2 + \left[\left(\frac{\omega}{r}\right)_{,r}\right]^2 -$$
$$- \frac{\lambda}{2r}\left[\left(\frac{\omega}{r}\right)^2\right]_{,r} + \frac{\omega_{,z}^2}{r^2}\Big\}.$$

With these expressions the energy equation, which determines the temperature, reads

$$\frac{3}{2}\varrho(\underline{u}\cdot\nabla)T + \varrho T\,\nabla\cdot\underline{u} - \nabla(\underline{\underline{k}}\,\nabla T) = \eta_\perp \underline{j}_\perp^2 + \eta_\parallel j_\parallel^2 + \underline{\underline{P}}:\nabla\underline{u}. \tag{58}$$

The condition of incompressibility can be derived from the energy equation. Equivalent to the energy relation (3) or (55) is the entropy equation

$$\varrho T\left(\frac{\partial S}{\partial t} + \underline{u}\cdot\nabla S\right) = \nabla\cdot(\underline{\underline{k}}\,\nabla T) + \underline{\underline{P}}:\nabla\underline{u} + \underline{j}\cdot(\underline{E} + \underline{u}\times\underline{B})$$

With $p = p(\varrho, S)$ we can easily write an equation for the pressure as

$$\frac{dp}{dt} = \frac{\partial p}{\partial \varrho}\frac{d\varrho}{dt} + \frac{\partial p}{\partial S}\frac{dS}{dt}$$

or

$$\frac{\partial p}{\partial t} + \underline{u}\cdot\nabla p + \frac{\partial p}{\partial \varrho}\varrho\,\nabla\cdot\underline{u} = \frac{\partial p}{\partial S}\left[\nabla\cdot\underline{\underline{k}}\,\nabla T + \underline{\underline{P}}:\nabla\underline{u} + \underline{j}\cdot(\underline{E} + \underline{u}\times\underline{B})\right]/\varrho T.$$

In the case of non-dissipate flow, where the right-hand side vanishes, we recover the usual conditions for the validity of incompressible flow, viz. the flow velocity must be small and the pressure variations must be small compared with the mean thermodynamic pressure. If dissipation is included, it follows that, in addition to the above conditions, the dissipation must not be so large that the right-hand side becomes comparable with the left. Physically, the dissipation-induced pressure

variations must be small compared with the other pressure variations in the system.

In a confiend plasma with substantial pressure variation, the assumption of incompressibility is not justified but is rather poor.

At the end of this section we summarize the set (S2) of compressible equations for axisymmetry:

(1) $$\rho\, \nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = \rho_s ,$$

(41) $$-j_0 \left[\eta_\perp + (\eta_\parallel - \eta_\perp)\frac{\chi^2}{r^2 \underline{B}^2}\right] + \frac{(\eta_\parallel - \eta_\perp)}{r^2 \underline{B}^2}\,\chi\,\nabla\psi\cdot\nabla\chi =$$
$$= r_0 E_0 - \underline{u}\cdot\nabla\psi + \eta_\parallel\,\nabla\psi\times\nabla\chi\cdot\nabla\theta ,$$

(42) $$\nabla\cdot\left(\frac{\chi}{r^2}\nabla\psi\right) + \nabla\psi\times\nabla\left(\frac{\omega}{r^2}\right)\cdot\nabla\theta =$$
$$= \nabla\cdot\left\{\frac{\eta_\perp}{r^2}\nabla\chi + \frac{\eta_\parallel - \eta_\perp}{r^2}\nabla\psi\,\frac{\eta_\perp\nabla\psi\cdot\nabla\chi + \chi\,(r_0 E_0 - \underline{u}\cdot\nabla\psi)}{\eta_\perp|\nabla\psi|^2 + \eta_\parallel\chi^2}\right. +$$
$$+ \eta_\parallel\,\frac{\nabla\theta\times\nabla\psi}{r^2}\,\frac{(r_0 E_0 - \underline{u}\cdot\nabla\psi + \eta_\parallel\nabla\theta\times\nabla\psi\cdot\nabla\chi)\,r^2\underline{B}^2}{\eta_\perp|\nabla\psi|^2 + \eta_\parallel\chi^2} +$$
$$+ \eta_\parallel\eta_\parallel\left(|\nabla\psi|^2 + \chi^2\right)\frac{\chi}{r^2}\,\frac{\nabla\chi\times\nabla\theta}{\eta_\perp|\nabla\psi|^2 + \eta_\parallel\chi^2}$$

(52) $$m\rho\left(\underline{u}\cdot\nabla u - \frac{\omega^2}{r^3}\right) + p_{,r} + \frac{1}{r^2}\left(\frac{\chi^2}{2}\right)_{,r} + j_0\,\frac{\psi_{,r}}{r^2} = F_r ,$$

(53) $$m\rho\,\underline{u}\cdot\nabla\omega + \nabla\psi\times\nabla\chi\cdot\nabla\theta = F_\theta ,$$

(54) $$m\rho\,\underline{u}\cdot\nabla v + \left(p + \frac{\chi^2}{2r^2}\right)_{,z} + j_0\,\frac{\psi_{,z}}{r^2} = F_z ,$$

(58) $$\frac{3}{2}\rho\,\underline{u}\cdot\nabla T + \rho T\,\nabla\cdot\underline{u} - \nabla(\underline{k}\,\nabla T) = \eta_\perp \underline{j}_\perp^2 + \eta_\parallel \underline{j}_\parallel^2 + \underline{\underline{P}}:\nabla\underline{u}$$
$$+ C_s .$$

With known velocity, the continuity equation is a hyperbolic equation for the density. For its solution appropriate sources have to be given. The remaining equations in the system (52) form, as discussed, an elliptic system, where boundary data can be prescribed.

For planar symmetry, the set of equations is of the form

(1') $$\rho\,\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = \rho_s,$$

(41') $$-j_0\,(\eta_\perp + (\eta_\parallel-\eta_\perp)\frac{\chi^2}{\underline{B}^2} = E_0 - \underline{u}\cdot\nabla\psi + \eta_\parallel\,\nabla\psi\times\nabla\chi\cdot\nabla z,$$

(42') $$\nabla(\eta_\perp\nabla\chi) + \nabla\psi\times\nabla\omega\cdot\nabla z =$$
$$= \nabla\cdot\Big\{\eta_\perp\nabla\chi + (\eta_\parallel-\eta_\perp)\nabla\psi\,\frac{\eta_\perp[\nabla\psi\cdot\nabla\chi + \chi(E_0-\underline{u}\cdot\nabla\psi)]}{\eta_\perp|\nabla\psi|^2+\eta_\parallel\chi^2} +$$
$$+ \eta_\parallel\,\nabla z\times\nabla\psi\,\frac{(E_0-\underline{u}\cdot\nabla\psi + \nabla z\times\nabla\psi\cdot\nabla\chi)\,\underline{B}^2}{\eta_\perp|\nabla\psi|^2+\eta_\parallel\chi^2}$$
$$+ \eta_\parallel\,\eta_\parallel(\chi^2+|\nabla\psi|^2)\chi\,\frac{\nabla\chi\times\nabla z}{\eta_\perp|\nabla\psi|^2+\eta_\parallel\chi^2}$$

(52') $$m\rho\,\underline{u}\cdot\nabla u + (p+\frac{\chi^2}{2})_{,x} + \psi_{,x}\,j_0 = \nabla\cdot\mu\nabla u + \frac{1}{3}(\mu\nabla\cdot\underline{u})_{,x} - \nabla\mu\times\nabla v\cdot\nabla z,$$

(53') $$m\rho\,\underline{u}\cdot\nabla v + (p+\frac{\chi^2}{2})_{,y} + \psi_{,y}\,j_0 = \nabla\cdot\mu\nabla v + \frac{1}{3}(\mu\nabla\cdot\underline{u})_{,y} + \nabla\mu\times\nabla u\cdot\nabla z,$$

(54') $$m\rho\,\underline{u}\cdot\nabla\omega + \nabla\psi\times\nabla\chi\cdot\nabla z = \nabla\cdot\mu\nabla\omega,$$

(58') $$\frac{3}{2}\rho\,\underline{u}\cdot\nabla T + \rho T\,\nabla\cdot\underline{u} - \nabla(\underline{k}\,\nabla T) =$$
$$= \eta_\perp\,\underline{j}_\perp^{\,2} + \eta_\parallel\,\underline{j}_\parallel^{\,2} + \mu\Big\{2(u_{,x}^2 + v_{,y}^2) - \frac{2}{3}(u_{,x}+v_{,y})^2 +$$
$$+ (v_{,x}+u_{,y})^2 + (\omega_{,x}^2+\omega_{,y}^2)\Big\} + Q_s$$

If there is no flow, i.e. $\underline{u} \equiv 0$, it follows from eq. (53) that $\chi = \chi(\psi)$ and eq. (51) reduces to the Grad-Shafranov equation.

In the case with purely toroidal flow, i.e. $u \equiv v \equiv 0$, it follows that $\underline{u} \cdot \nabla = 0$ and $\nabla \cdot \underline{u} = 0$. The continuity equation then does not determine the density. Equations (41), (42) and (53) form a closed set for ψ , χ and ω . The two remaining poloidal components of the momentum equation determine the thermodynamic quantities ϱ and p . As discussed for incompressible flow, in the case with viscosity, the momentum imbalance may act as a source for poloidal rotation even if no rotation is imposed at the wall.

For zero viscosity, $\mu \equiv 0$, the system has quite different properties. The momentum equation reduces to

$$m\varrho\, \underline{u} \cdot \nabla \underline{u} = -\nabla p - \frac{j_0}{r^2} \nabla\psi - \frac{\chi}{r^2} \nabla\chi + \nabla\theta\, \nabla\chi \times \nabla\psi \cdot \nabla\theta .$$

For given ψ one has to solve a system of mixed type with sonic transition, which causes difficulties as discussed in Sect. II.

IV. General Pressure Tensor

In the analysis so far a simplified pressure tensor has been taken into account. A strong magnetic field, however, requires a more general tensor than that from ordinary fluid theory. This tensor is derived from microscopic theory; see, for example, Braginskii / 5 /.

In our opinion it is useful to formulate the stress-strain relations in a manner independent of the origin of the viscous forces, but dependent only on covariance properties and simple physical bypotheses. In particular, we assume that the stresses are linear functionals of the strain matrix $\underline{\underline{S}}$:

$$S_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad . \tag{59}$$

We can generalize this treatment to allow the coefficients of the linear relation to depend on the invariants of that form, namely on the traces Tr $\underline{\underline{S}}$, Tr $\underline{\underline{S}}^2$ and Tr $\underline{\underline{S}}^3$. Additionally, we assume that the stresses can depend on the polar vector $\underline{B}$, but on no other quantities. Again, the coefficients in the linear stress-strain relation can be generalized to depend on the invariants found from $\underline{\underline{S}}$ and $\underline{B}$. The stress matrix $\underline{\underline{\mathbb{P}}}$ must be symmetric and invariant under inversion of coordinates, since $\underline{B}$ is not. We shall construct the corresponding form and show that it is essentially equivalent to that of Braginskii with arbitrary coefficients. We shall adjoin three additional hypotheses. We postulate that the viscous forces must increase the entropy of the system and we assume that the resulting viscous equations must generate an elliptic stress $\nabla \cdot \underline{\underline{\mathbb{P}}}$. Finally, for convenience, but not essentially, we assume that similar to an ordinary fluid there is no bulk viscosity, or that the viscous stress only depends on

$$W_{ij} = S_{ij} - \frac{1}{3}\delta_{ij} \operatorname{Tr} \underline{\underline{S}} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}(\nabla \cdot \underline{u}) \tag{60}$$

It is convenient to employ the vector $\underline{B}$ in two distinct forms. We use $\underline{B}$ interchangeably as a row or column vector and we introduce the standard matrix from electromagnetic theory.

$$\underline{\underline{B}} = \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix}, \tag{61}$$

which is invariant under inversion of coordinates.
For any vector $\underline{V}$ it holds that

$$\underline{\underline{B}}\,\underline{V} = \underline{B} \times \underline{V},$$
$$\underline{V}\,\underline{\underline{B}} = \underline{V} \times \underline{B},$$

while

$$\underline{\underline{B}} \cdot \underline{B} = \underline{B} \cdot \underline{\underline{B}} = 0,$$

and

$$\underline{\underline{B}} \cdot \underline{\underline{B}} = \underline{B}\,\underline{B} - \underline{B}^2\,\underline{\underline{I}}.$$

The stress tensor must be even in $\underline{B}$ and of any order in $\underline{\underline{B}}$.

Although we shall drop the bulk viscosity, for completeness we give the most general stress tensor proportional to ($\nabla \cdot \underline{u}$). The most general form consistent with the covariance properties is clearly

$$\nabla \cdot \underline{u}\ \left(\mu_1^b\,\underline{\underline{I}} + \mu_2^b\,\underline{B} \cdot \underline{B} \right),$$

where μ_1^b and μ_2^b are the two bulk viscosity coefficients, which we set equal to zero.

We construct the stress tensor from W_{ij} , $\underline{B}$ and $\underline{\underline{B}}$.
We systematize the procedure by taking terms of increasing degree in $\underline{B}$ and $\underline{\underline{B}}$. Clearly, the only term of degree zero is

$$\mu_0 \underline{\underline{W}},$$

where μ_0 is the first viscous coefficient. As indicated above, we might allow μ_0 to be a function of $\underline{B}^2$. Any term linear in $\underline{B}$ or $\underline{\underline{B}}$ must be linear in $\underline{\underline{B}}$ since $\underline{B}$ is a polar vector. Such a term is $\underline{\underline{B}}$ $\underline{\underline{W}}$ and symmetry requires that we take the symmetric form

$$\mu_1\left(\underline{\underline{W}}\,\underline{\underline{B}} - \underline{\underline{B}}\,\underline{\underline{W}}\right)$$

The only terms quadratic in $\underline{B}$ and $\underline{\underline{B}}$ are $\underline{\underline{B}}$ $\underline{\underline{W}}$ $\underline{\underline{B}}$, $\underline{B}$ $\underline{B}\cdot\underline{\underline{W}}$ + $\underline{\underline{W}}\cdot\underline{B}\,\underline{B}$ and $\underline{\underline{\Pi}}$ $\underline{B}\cdot\underline{\underline{W}}\cdot\underline{B}$. A direct calculation shows that

(62) $-\underline{B}^2\,\underline{\underline{W}} + \underline{\underline{B}}\,\underline{\underline{W}}\,\underline{\underline{B}} + \underline{B}\,\underline{B}\cdot\underline{\underline{W}} + (\underline{\underline{W}}\cdot\underline{B})\,\underline{B} = \underline{\underline{\Pi}}\,\underline{B}\cdot\underline{\underline{W}}\cdot\underline{B}$,

so that if we allow μ_0 to depend on B^2, we may eliminate one of these terms. We thus take the two quadratic terms

$$\mu_2\,\underline{\underline{B}}\,\underline{\underline{W}}\,\underline{\underline{B}} + \mu_3\,\underline{\underline{\Pi}}\,\underline{B}\cdot\underline{\underline{W}}\cdot\underline{B}$$

The cubic terms must have either two factors of $\underline{B}$ or none. If there are no factors of $\underline{B}$, the form must be $\underline{\underline{B}}^3\cdot\underline{\underline{W}}$, $\underline{\underline{B}}^2$ $\underline{\underline{W}}$ $\underline{\underline{B}}$, $\underline{\underline{B}}$ $\underline{\underline{W}}$ $\underline{\underline{B}}^2$ or $\underline{\underline{W}}$ $\underline{\underline{B}}^3$. The identity $\underline{\underline{B}}^2 = \underline{B}\,\underline{B} - \underline{B}^2\,\underline{\underline{\Pi}}$ then reduces the form to the first case. There must thus be two factors of $\underline{B}$ and one factor of $\underline{\underline{B}}$. The only such form is

$$\mu_4\left(\underline{\underline{B}}\,\underline{B}\cdot\underline{\underline{W}}\,\underline{B} - \underline{B}\cdot\underline{\underline{W}}\cdot\underline{B}\,\underline{\underline{B}}\right)$$

Similar reduction arguments show that there is only one new term of fourth degree and no terms of higher degree

$$\mu_5\left(\underline{B}\,\underline{B}\,\underline{B}\cdot\underline{\underline{W}}\cdot\underline{B}\right)$$

We have thus derived the result for the stress tensor:

$$\text{(63)}\qquad \underline{\underline{P}} = \mu_0\,\underline{\underline{W}} + \mu_1\left(\underline{\underline{W}}\,\underline{\underline{B}} - \underline{\underline{B}}\,\underline{\underline{W}}\right) + \mu_2\,\underline{\underline{B}}\,\underline{\underline{W}}\,\underline{\underline{B}} + $$
$$+ \mu_5\,\underline{\underline{\Pi}}\,\underline{B}\cdot\underline{\underline{W}}\cdot\underline{B} + \mu_4\left(\underline{\underline{B}}\,\underline{B}\cdot\underline{\underline{W}}\cdot\underline{B} - \underline{B}\cdot\underline{\underline{W}}\cdot\underline{B}\,\underline{\underline{B}}\right) +$$
$$+ \mu_5\,\underline{B}\,\underline{B}\;\underline{B}\cdot\underline{\underline{W}}\cdot\underline{B}\,.$$

A direct calculation shows that

$$\mathrm{Tr}\,\underline{\underline{P}} = \underline{B}\cdot\underline{\underline{W}}\cdot\underline{B}\left\{\mu_2 + 3\mu_3 + \underline{B}^2\mu_5\right\}$$

The viscous entropy production is

$$\underline{\underline{P}} : \nabla \underline{u} = \frac{1}{2} \underline{\underline{P}} : \underline{\underline{S}} = \frac{1}{2} \underline{\underline{P}} : \underline{\underline{W}} + \frac{1}{3} \mathrm{Tr}\, \underline{\underline{P}} (\nabla \cdot \underline{u})$$

and positivity requires that the coefficient of $\nabla \cdot \underline{u}$ vanish

$$\mu_2 + 3\mu_3 + \underline{B}^2 \mu_5 = 0. \tag{64}$$

In this case we have

$$\underline{\underline{P}} : \underline{\underline{W}} = \mathrm{Tr}\, (\underline{\underline{P}}\,\underline{\underline{W}}) = \mu_0 \,\mathrm{Tr}\, \underline{\underline{W}}^2 + \mu_2 \,\mathrm{Tr}\, (\underline{\underline{B}}\,\underline{\underline{W}})^2 + \\ + \mu_5 \,(\underline{B} \cdot \underline{\underline{W}} \cdot \underline{B})^2$$

We must assume that for all symmetric, trace-free matrices $\underline{\underline{W}}$

$$\mu_0 \,\mathrm{Tr}\, \underline{\underline{W}}^2 + \mu_2 \,\mathrm{Tr}\, (\underline{\underline{B}}\,\underline{\underline{W}})^2 + \mu_5 \,(\underline{B} \cdot \underline{\underline{W}} \cdot \underline{B})^2 > 0 . \tag{65}$$

If we orientate $\underline{B}$ in the z direction, it is easy to compare our form with that of Braginskii. A direct calculation shows that

$$\eta_0 = \mu_0 - \underline{B}^2 (\mu_2 + 2\mu_5) ,$$

$$\eta_1 = \mu_0 + \underline{B}^2 \mu_2 ,$$

$$\eta_2 = \mu_0 ,$$

$$\eta_3 = 2\mu_1 |\underline{B}| ,$$

$$\eta_4 = (\mu_1 + \mu_4 \underline{B}^2) |\underline{B}| ,$$

so that the constraint eq. (64) determines μ_5 . Braginskii with general coefficient η_i is thus the most generally admissible form for the stress tensor.

The last question we tackle, albeit incompletely, is the ellipticity of the viscous operator $\nabla\cdot\underline{\underline{\mathbb{P}}}$. One might expect that positivity of the entropy production, eq. (65), is equivalent to ellipticity. We can readily show that this is not the case although the relation between the two conditions is as yet open. We examine the ellipticity of the operator in the full three-dimensional case. Ellipticity requires that given any surface $\varphi(x,y,z) = 0$ with the normal vector $\underline{k} = \nabla\varphi$ and $\underline{u}$ and $(\underline{k}\cdot\nabla)\,\underline{u} = \partial\underline{u}/\partial\varphi$ on the surface, one solves for the second derivative $(\underline{k}\cdot\nabla)^2\,\underline{u} = \partial^2\underline{u}/\partial\varphi^2$. Clearly, if we can solve for the normal derivatives in the general case, we can also solve under the restriction of axisymmetry. We note that hyperbolicity or parabolicity of an operator need not persist as symmetry is assumed. To decide ellipticity, it suffices to take $\underline{B} = (0,0,B)$ and orientate the x direction so that $\underline{k} = (k_x, k_y, 0)$. We define $\partial^2\underline{u}/\partial\varphi^2 = (\tilde{u}, \tilde{v}, \tilde{w})$, and we also assume that $\underline{B}$ and μ are constant. A direct calculation shows that

$$(\nabla\cdot\mathbb{P})_x = \left[\tfrac{2}{3}k_x^2\,(2\mu_0 + \hat{\mu}_2 - \hat{\mu}_3) + \mu_0 k_y^2\right]\tilde{u} + \left[2\mu_1 k_x^2 + (\hat{\mu}_1 + \mu_4)\,k_y^2\right]\tilde{v}$$

$$+ \tfrac{1}{3}k_x k_y\,(\mu_0 + 2\hat{\mu}_2 + 4\mu_3)\,\tilde{w}\,,$$

$$(\nabla\cdot\mathbb{P})_y = -\left[2k_x^2\hat{\mu}_1 + k_y^2\,(\hat{\mu}_1 + \hat{\mu}_4)\right]\tilde{u} + \left[k_x^2(\mu_0 + \hat{\mu}_2) + k_y^2\mu_0\right]v\; -$$

$$- k_x k_y\,(\hat{\mu}_1 + \hat{\mu}_4)\,\tilde{w}\,,$$

$$(\nabla\cdot\mathbb{P})_z = \tfrac{1}{3}k_x k_y\,(\mu_0 + 2\hat{\mu}_2 + 4\hat{\mu}_3)\,\tilde{u} + (\hat{\mu}_1 + \hat{\mu}_4)\,k_x k_y\,\tilde{v}\; +$$

$$+ \left[\mu_0 k_x^2 + \tfrac{4}{3}(\mu_0 - \hat{\mu}_2 - 2\hat{\mu}_3)\,k_y^2\right]\tilde{w}$$

where $\hat{\mu}_1 = \mu_1 B$, $\hat{\mu}_2 = \mu_2 B^2$, $\hat{\mu}_3 = \mu_3 B^2$ and $\hat{\mu}_4 = \mu_4 B^3$.

The necessary and sufficient condition for ellipticity is that the above system be solvable for $\tilde{u}, \tilde{v}$ and $\tilde{w}$ for any $\underline{k} \neq 0$ or that the associated matrix of coefficients be invertible for any $\underline{k} \neq 0$. Hence it is required that the determinant Δ does not vanish for any $k \neq 0$ where

$$\Delta = \det \begin{vmatrix} \frac{2}{3}k_x^2(2\mu_0+\hat{\mu}_2-\hat{\mu}_3)+\mu_0 k_y^2 & 2\hat{\mu}_1 k_x^2+(\hat{\mu}_1+\hat{\mu}_4)k_y^2 & \frac{1}{3}k_x k_y(\mu_0+2\hat{\mu}_2+4\hat{\mu}_3) \\ -2\hat{\mu}_1 k_x^2-(\hat{\mu}_1+\hat{\mu}_4)k_y^2 & k_x^2(\mu_0+\hat{\mu}_2)+k_y^2\mu_0 & -k_x k_y(\hat{\mu}_1+\hat{\mu}_4) \\ \frac{1}{3}k_x k_y(\mu_0+2\hat{\mu}_2+4\hat{\mu}_3) & k_x k_y(\hat{\mu}_1+\hat{\mu}_4) & \mu_0 k_x^2+\frac{4}{3}(\mu_0-\hat{\mu}_2-2\hat{\mu}_3)k_y^2 \end{vmatrix}$$

If a factor (-1) is taken out of the second row, then the matrix is symmetric. The condition that Δ does not vanish, becomes a homogeneous polynomial of degree 3 in the variables (k_x^2) and (k_y^2). If one introduces the variable $\alpha=(k_y/k_x)^2$, then $\Delta \neq 0$ is equivalent to non-existence of positive roots of a polynomial of degree three in α. One can write down conditions on the coefficients of a polynomial to satisfy those conditions. It is easy to see that the condition involves μ_1 , and μ_4 , coefficients missing in the condition for positivity of entropy production. The ellipticity condition is thus different from the entropy condition, but we do not know which is more stringent.

It is of some interest to examine the ellipticity condition and its implications for the case of Braginskii transport. We examine the condition $\Delta > 0$ solely under the assumptions that the coefficients scale an appropriate manner in the small parameter $\epsilon=(\omega_{c_i}\cdot\tau_i)^{-1}$ and that certain positivity conditions hold. For our purposes the precise interpretation of ϵ and $(\omega_{c_i}\tau_i)^{-1}$ is not essential, just so long as ϵ is small. Hence, neoclassical transport, which has similar scaling in a small parameter ϵ , is also included in our treatment. We write all the transport coefficients as multiples of the parallel viscosity η_0

$$\begin{aligned} 2\hat{\mu}_3 &= -\eta_0\nu_3 \\ 2\hat{\mu}_1 &= \epsilon\eta_0\nu_1 \\ \hat{\mu}_1+\hat{\mu}_4 &= \epsilon\eta_0\nu_4 \\ \mu_0 &= \epsilon^2\eta_0\nu_0 \\ \mu_0+\hat{\mu}_2 &= \epsilon^2\eta_0\nu_2 \end{aligned}$$

With these definitions ν_i, $i=0,1,\ldots,4$ are all pure numbers of order 1 in ϵ and in Braginskii transport ν_0 , ν_2 , ν_3 are positive, while ν_1 and ν_4 are of the same sign, which is positive for ions and negative for electrons. Presumably, neoclassical transport has similar scaling and positivity properties. It is then elementary to show that

$$\Delta = \epsilon^2 \eta_0^3 \left\{ \frac{2}{3} \nu_3 k_y^2 \left[(2\nu_1 + \nu_4) k_x^2 + 2\nu_4 k_y^2 \right]^2 + \right.$$

$$\left. + \epsilon^2 \left[k_x^2 \nu_0 \left(\frac{2}{3} \nu_3 (\nu_0 + \nu_2) + \nu_1^2 \right) + k_y^2 P(k_x, k_y, \epsilon) \right] \right\}$$

where $P(k_x, k_y, \epsilon)$ is a homogeneous polynomial of degree four in k_x and k_y whose coefficients are polynomials in ϵ .

The first term in the expression for delta is non-negative, but vanishes for $k_y = 0$. For k_y of order ϵ or smaller the second term is of the order of the first term but strictly positive. Thus, for ϵ small $\Delta > 0$ for $k_x^2 + k_y^2 \neq 0$. Hence the viscous operator is strictly elliptic for coefficients that obey the scaling and positivity assumptions.

We can infer other information on the nature of the equations from the form of Δ . If we ignore terms of order $\hat{\epsilon}$ in Δ , then Δ is non-negative, but vanishes for $k_y = 0$, or for surfaces such that $(\underline{B} \cdot \nabla) \varphi = 0$ For the approximate operator flux surfaces are characteristic surfaces and there is some hyperbolic character to the system. When we add the full dissipation ellipticity occurs. Thus, the dissipative operator is of the form $[\lambda (\underline{B} \times \nabla)^2 + \mu (\underline{B} \cdot \nabla)^2]^2 ((\underline{B} \nabla)^2 + \epsilon^2 \nu (\underline{B} \times \nabla)^2)$ for positive λ, μ, ν .

It is tempting to consider omitting the small term that guarantees ellipticity, which we could accomplish by keeping μ_1, μ_3, and μ_4 (or η_0, η_3 and η_4 in Braginskii notation) and dropping the other coefficients.It is not at all clear that this procedure would be satisfactory. For those flux surfaces which are closed and do not meet the boundary, we must give some data. Thus, we are not likely to be able to prescribe the full viscous boundary data $\underline{u} = 0$, and we are again required to give a profile in the plasma. If we hope to give plasma sources and boundary data and infer conditions inside the plasma, then we will fail. In order to generate the full elliptic character, we need all the viscous coefficients, although we could indeed drop some terms in the viscous operator that are small and do not affect the ellipticity.

At this point we are unprepared to propose an approximate system to represent the viscous flow instead of the full set. We are certainly aware that the smallness of the viscous effects, which we examine further in the next section, makes the direct numerical solution of the full problem exceedingly difficult. Despite the difficulty, the direct numerical solution would be the clearest option. If this is not feasible, then various other approaches are possible. One could generalize "1 1/2 D" transport to the case of weak dissipation superimposed on a steady state with toroidal flow. Any approximation must take into consideration the basic nature of the original equations and we have layed out the underlying structure here.

V. Transport Coefficients

The macroscopic fluid equations require the knowledge of the transport coefficients which are obtained from kinetic theory. In the collisional regime the transport coefficients are known. The coefficients are expressed in terms of collision time τ_e or τ_i for electrons and ions respectively /5 /

$$\tau_e = \frac{3\sqrt{m_e}\, T_e^{3/2}}{4\sqrt{2\pi}\,\lambda e^4 \varrho_e} ,$$

where m_e is the electron mass, λ the Coulomb logarithm, e the electron charge and ϱ_e the electron gyroradius, and respectively

$$\tau_i = \frac{3\sqrt{m_i}\, T_i^{3/2}}{4\sqrt{\pi}\,\lambda e^4 \varrho_i} .$$

The ratio of the different collision times is for equal temperature

$$\frac{\tau_i}{\tau_e} = \sqrt{\frac{2 m_i}{m_e}} .$$

The gyrofrequency is

$$\omega_{c_e} = \frac{eB}{m_e C}$$

$$\omega_{c_i} = \frac{eB}{m_i C}$$

with C denoting the speed of light.
Their ratio is

$$\frac{\omega_{c_i}}{\omega_{c_e}} = \frac{m_e}{m_i}$$

The factor $\omega_{c_i} \cdot \tau_i$ has a large value $\omega_{c_i} \cdot \tau_i \approx 10^3 - 10^6$, for typical tokamak parameters $\omega_{c_i} \cdot \tau_i \approx 10^5$.

The resistivity is given by Spitzer's formula

(66a) $$\eta_\perp^{-1} = \sigma_\perp = \frac{e^2 n_e \tau_e}{m_e} \propto T_e^{3/2}$$

and

(66b) $$\eta_\parallel^{-1} = \sigma_\parallel = 1.96\, \sigma_\perp$$

The Hall term has the form

$$\eta_H = \frac{1}{e n_e c} \quad .$$

The heat flux is governed by the ions

(67)
$$\kappa_\parallel = 3.9 \frac{n_i T_i \tau_i}{m_i} \quad ,$$

$$\kappa_\perp = \frac{2}{3.9} \kappa_\parallel \frac{1}{(\omega_{c_i} \cdot \tau_i)^2} \quad ,$$

$$\kappa_\wedge = \frac{5}{2 \cdot 3.9} \kappa_\parallel \frac{1}{\omega_{c_i} \cdot \tau_i} \quad .$$

The conductivity is much larger along the magnetic field than perpendicular to it, namely by a factor of $(\omega_{c_i} \cdot \tau_i)^2$ which ranges between 10^6 and 10^{12}.

The fluid model reveals the known Pfirsch-Schlüter factor /4 / if an average over flux surfaces is performed. For example, the perpendicular heat conductivity is then

$$\bar{\kappa}_\perp = \kappa_\perp (1 + 1.6\, q^2) \quad ,$$

where q denotes the safety factor.

The viscosity is related with the ions. The factors given by Braginskii / 5 / read

(68)
$$\eta_0 = 0.96\, n_i \tau_i T_i$$
$$\eta_1 = \frac{0.3}{0.96}\, \eta_0 / (\omega_i \cdot \tau_i)^2$$
$$\eta_2 = 4\, \eta_1$$
$$\eta_3 = \frac{0.5}{0.96}\, \eta_0 / (\omega_i \cdot \tau_i)$$
$$\eta_4 = 2\, \eta_3$$

Again the parallel viscosity η_0 is much larger than the perpendicular ones.
These transport coefficients increase with temperature.
If the temperature is sufficiently high, this model is no longer valid. The trapping of the particles in bananas is taken into account in the neoclassical theory. The coefficients in this regime are usually obtained by surface averaging.
In the transport code of ref. / 6 /, the corrected expressions are

$$\eta_\perp^{NC} = \eta_\perp / f_{tr} \quad ,$$

where f_{tr} represents the trapping correction, eq. (5) in ref. / 6 /.
The ion thermal conductivity contains the neoclassical contribution

$$\bar{\kappa}_\perp^{NC} = \frac{0.68}{1 + 0.36\, \nu_i^*} \; \frac{n_{\vartheta i_0}^2}{\tau_i} \left(\frac{r}{R}\right)^{1/2} + \bar{\kappa}_\perp$$

where the factors are given in eq. (7) of ref. / 6 /.
In this fashion corrected transport coefficients valid in various regions of collisionality can be found in the literature. In the derivation, however, many assumptions and orderings are made, especially the plasma flow is assuemd to be small compared with the thermal speed. In this paper we want to examine especially cases with a large flow speed. It is possible that such flows change the local electric field, a quantity essential for the transport, thus yielding different results. Therefore the transport coefficients needed for our fluid model are not known in detail. At present we have found to choose best approximation as discussed above.
If anomalous transport is to be included, the factors have to be changed accordingly in a phenomenological way, e.g. in form of an anomalous resistivity.
An interesting aspect of the equilibrium model is, therefore, its use to compute better transport coefficients. A simple way would be to insert the parameters from experiments and then to study the consequences and to correct them until a steady state is found. More interesting and challenging is it to use the macroscopic equilibria with large flow, which can be computed, and to calculate the transport following the procedure used in the review paper of Hirschmann and Sigmar / 7/ with appropriate ordering.

VI. Discussion

The resistive, viscous fluid MHD equations are the basis for describing tokamak behaviour as an equilibrium problem. This leads to a set of non-linear, two-dimensional equations. Their numerical approximation by means of the finite-element method is a complex but feasible task.
The good results with quadrilateral, biquadratic elements used for the ideal MHD equations with flow, Ref. /2/, prompt their application for the model presented. The successful use of the finite-element method, especially for elliptic equilibrium problems with complex domain and, in particular, the application to the Navier-Stokes equation yields additional guidance.
The system contains typically eight scalar quantities which have to be determined iteratively. We do not discuss special possible iteration schemes here, but discuss the general properties of the system of equations.

The incompressible, resistive, MHD model with or without viscosity is very appealing in view of its simplicity, but its validity is questionnable. As already indicated, for an incompressible approximation to be good, the pressure variation must be small compared with the mean pressure. Low-speed aerodynamics or water flows are typical appropriate systems. In tokamak plasmas the pressure variation is of the order of 100 %. Another inconvenience in the model is the difficulty of introducing sources. The advantage of the assumption of incompressibility is that one can introduce a stream function and then define the mass density as a function of the stream function. The use of the stream function permits simpler sets of equations. However, in many problems one wants to include distributed mass and momentum sources. It is difficult to admit a distributed mass source and employ a stream function. Without the use of a stream function there is little advantage in using incompressible equations rather than compressible equations. Just as in fluid dynamics, there is a great difference between viscous and inviscid flows. In the former case the equations are everywhere elliptic, while in the latter the equations are hyperbolic or elliptic of lower order and possess sonic transitions.

The viscous, compressible resistive MHD model appears complex but well within the capability of present computation. It is easy to allow distributed sources and sources at the boundaries. This model lacks some of the physics of plasmas that one would like to include, such as full treatment of electron and ion species. The Ohm's law reveals this approximation in neglecting the pressure gradients and the electron viscosity. For the treatment of anomalous transport the inclusion of the electron viscosity is necessary since this diffusion is more likely caused by an anomalous parallel electron viscosity than by an anomalous perpendicular resistivity (see ref. /8/). But the simplicity of the model used makes up for its lack of physical completeness. We have started an examination of the local classical or approximated neoclassical transport coefficients. The system of equations with the specification of boundary data and source terms, as discussed in the paper, is well defined and solvable. A major difficulty results from the fact that the transport coefficients are grossly disparate in magnitude. A value of $\omega_{ci} \cdot \tau_i \sim 10^3$ appears tolerable. But a value of $\omega_{ci} \cdot \tau_i \sim 10^5$ introduces highly different scales in the elliptical momentum equation. It is questionable whether the very small perpendicular diffusion can be determined simultaneously with the large flow within a magnetic surface. It is possible that the smaller perpendicular coefficients do not contribute. Then the solution would be identical to one resulting from a reduced system without resistivity and conductivity. The viscosity should be included to keep the equation in elliptical form. It is not clear whether the viscosity acts in the plasma only in boundary layers and in the energy equation or in the entire domain. Starting from such a reduced equilibrium, the perpendicular flow is evaluated by averaging over surfaces which takes care of the cancellation of large flows. Then an generalization of the 1 1/2 dimensional transport in a system with flow arises. This possibility will be examined subsequently.

Although the conventional viscous tensor is very different from the viscous tensor, its simplicity is a strong argument for its use in preliminary studies aiming at simulation of anomalous transport.

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Axisymmetric, non-ideal
MHD states with
steady flow.